Lie Groups

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We consider Lie groups of matrices. In Lie groups it is possible to get to any element from the identity element by continuous variation of the parameters in the parameterization. First we must pick a parameterization. Take for example GL(2,R). This is a four-parameter group which we can parameterize with $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ as follows.

$$A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix} = \begin{pmatrix} 1+\alpha_1 & \alpha_2 \\ \alpha_3 & 1+\alpha_4 \end{pmatrix}$$

Other groups will usually have more restrictions on the group elements and so will have less than n^2 parameters, where n is the degree of the matrices.

We may represent elements that are sufficiently close to the identity element by their first order Taylor expansion. Let r be the number of parameters of the group.

$$A(d\boldsymbol{\alpha}) = A(\mathbf{0}) + \sum_{i=1}^{r} d\alpha_i \left(\frac{\partial A}{\partial \alpha_i} \right)\bigg|_{\alpha_i=0}$$

So we define the generator matrices[1]

$$X_i = \left(\frac{\partial A}{\partial \alpha_i} \right)\bigg|_{\alpha_i=0}$$

Thus we can write any matrix sufficiently close to identity as (using Einstein summation notation)

$$A(d\boldsymbol{\alpha}) = A(\mathbf{0}) + d\alpha_i X_i$$

We use the 2-dimensional rotation group, SO(2) as an example. The condition that the determinant is 1 provides one constraint and the orthogonality condition provides two constraints (row vectors perpendicular and column vectors perpendicular), so this is a $2^2 - 1 - 2 = 1$ parameter group. The canonical parameterization is

$$A(\alpha_1) = \begin{pmatrix} \cos(\alpha_1) & -\sin(\alpha_1) \\ \sin(\alpha_1) & \cos(\alpha_1) \end{pmatrix}$$

Therefore

$$X_1 = \left(\frac{\partial A}{\partial \alpha_1} \right)\bigg|_{\alpha_1=0} = \begin{pmatrix} -\sin(\alpha_1) & -\cos(\alpha_1) \\ \cos(\alpha_1) & -\sin(\alpha_1) \end{pmatrix}\bigg|_{\alpha_1=0} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

We can now perform any infinitesimal rotation by an angle $d\alpha_1$ using the first order Taylor expansion.

$$\left(\begin{smallmatrix} x' \\ y' \end{smallmatrix}\right) = A(d\alpha_1) \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = (A(\mathbf{0}) + d\alpha_1 X_1) \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + d\alpha_1 \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right] \begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \begin{pmatrix} x \\ y \end{pmatrix} + d\alpha_1 \begin{pmatrix} -y \\ x \end{pmatrix} = \begin{pmatrix} x - y\,d\alpha_1 \\ y + x\,d\alpha_1 \end{pmatrix}$$

The matrix generators can be replaced with equivalent operator generators. Using these operator generators we can rewrite the first order Taylor expansion completely in terms of operators. The identity matrix becomes the identity operator, 1.

$$\hat{A}(d\boldsymbol{\alpha}) = 1 + d\alpha_i \hat{X}_i$$

In the case of SO(2) we have

$$X_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \leftrightarrow \hat{X}_1 = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$$

[1] Question: Should it be the whole α vector gets set to zero?

To see the equivalence, we again perform a rotation by an infinitesimal angle $d\alpha_1$,

$$
\begin{aligned}
\binom{x'}{y'} &= \hat{A}(d\alpha_1)\binom{x}{y} = (1 + d\alpha_1 \hat{X}_1)\binom{x}{y} \\
&= \left(1 + d\alpha_1\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\right)\begin{pmatrix} x \\ y \end{pmatrix} \\
&= \begin{pmatrix} x \\ y \end{pmatrix} + d\alpha_1\left(x\begin{pmatrix} 0 \\ 1 \end{pmatrix} - y\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) \\
&= \begin{pmatrix} x - y\,d\alpha_1 \\ y + x\,d\alpha_1 \end{pmatrix}
\end{aligned}
$$

We obtain the same answer as we did when the generators were matrices, which shows their equivalence.

It is possible to create a matrix for finite transformations from the matrix for infinitesimal transformations. Write the infinitesimal parameters as $d\alpha_i = \frac{\alpha_i}{N}$ where N is an arbitrarily large number. Then perform the transformation N times and take the limit as N goes to infinity.

$$
A(\boldsymbol{\alpha}) = \lim_{N\to\infty}\left(A(\mathbf{0}) + \frac{\alpha_1}{N}X_i\right)^N = \exp(\alpha_i X_i)
$$

This is still a matrix where the exponential of a matrix is defined by the Taylor expansion.