Electromagnetic potential continuity equation

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A derivation of Maxwell's equations in potential form from Maxwell's equations in differential form. At the end, a continuity equation for the electromagnetic potentials is identified and discussed.

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ELECTROMAGNETIC FIELDS IN TERMS OF POTENTIALS

Starting from Maxwell's equations in differential form,

$$\begin{aligned}\nabla\cdot\mathbf{E} &= \frac{\rho}{\epsilon_0}\\ \nabla\times\mathbf{E} &= -\frac{\partial\mathbf{B}}{\partial t}\\ \nabla\cdot\mathbf{B} &= 0\\ \nabla\times\mathbf{B} &= \mu_0\mathbf{J}+\mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t}\end{aligned}$$

Maxwell's equations in potential form can be derived using Helmholtz's theorem.

According to Helmholtz's Theorem, any vector field $\mathbf{B}$ satisfying $\nabla\cdot\mathbf{B}=0$ can be written as

$$\mathbf{B}=\nabla\times\mathbf{A}$$

for some vector field $\mathbf{A}$. Based on Gauss's Law for magnetism (3rd equation), this theorem applies to the magnetic field. Plugging this into Faraday's law (2nd equation) gives

$$\nabla\times\mathbf{E}=-\frac{\partial}{\partial t}\left(\nabla\times\mathbf{A}\right)=-\nabla\times\left(\frac{\partial\mathbf{A}}{\partial t}\right)$$

$$\nabla\times\left(\mathbf{E}+\frac{\partial\mathbf{A}}{\partial t}\right)=0$$

Again by Helmholtz's Theorem, any vector field $\mathbf{F}$ satisfying $\nabla\times\mathbf{F}=0$ can be written as

$$\mathbf{F}=\nabla\phi$$

for some scalar field ϕ. Based on the last result, we can write

$$\mathbf{E}+\frac{\partial\mathbf{A}}{\partial t}=-\nabla\phi$$

$$\mathbf{E}=-\nabla\phi-\frac{\partial\mathbf{A}}{\partial t}$$

Therefore, the electromagnetic fields expressed in terms of potentials are

$$\mathbf{B}=\nabla\times\mathbf{A}$$

$$\mathbf{E}=-\nabla\phi-\frac{\partial\mathbf{A}}{\partial t}$$

MAXWELL'S EQUATIONS IN POTENTIAL FORM

Maxwell' s equations in potential form are obtained by eliminating $\mathbf{E}$ and $\mathbf{B}$ from the remaining two Maxwell equations using the substitutions derived in the previous section. Gauss's Law (1st equation) becomes

$$\nabla\cdot\left(-\nabla\phi-\frac{\partial\mathbf{A}}{\partial t}\right)=\frac{\rho}{\epsilon_0}$$

$$\nabla^2\phi+\frac{\partial}{\partial t}\left(\nabla\cdot\mathbf{A}\right)=-\frac{\rho}{\epsilon_0}$$

Ampere's Law (4th equation) becomes

$$\nabla\times\left(\nabla\times\mathbf{A}\right)=\mu_0\mathbf{J}+\mu_0\epsilon_0\frac{\partial}{\partial t}\left(-\nabla\phi-\frac{\partial\mathbf{A}}{\partial t}\right)$$

Using the relation $\mu_0\epsilon_0=1/c^2$ and the vector identity $\nabla\times(\nabla\times\mathbf{A})=\nabla(\nabla\cdot\mathbf{A})-\nabla^2\mathbf{A}$,

$$\nabla(\nabla\cdot\mathbf{A})-\nabla^2\mathbf{A}=\mu_0\mathbf{J}-\frac{1}{c^2}\nabla\left(\frac{\partial\phi}{\partial t}\right)-\frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2}$$

$$\nabla^2\mathbf{A}-\frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2}-\nabla\left(\nabla\cdot\mathbf{A}+\frac{1}{c^2}\frac{\partial\phi}{\partial t}\right)=-\mu_0\mathbf{J}$$

So Maxwell's equations in potential form can be written in the following form

$$\nabla^2\phi+\frac{\partial}{\partial t}\left(\nabla\cdot\mathbf{A}\right)=-\frac{\rho}{\epsilon_0}$$

$$\nabla^2\mathbf{A}-\frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2}-\nabla\left(\nabla\cdot\mathbf{A}+\frac{1}{c^2}\frac{\partial\phi}{\partial t}\right)=-\mu_0\mathbf{J}$$

GAUGE FIXING

Both of the equations derived in the previous section contain the expression $\nabla\cdot\mathbf{A}$. The value of this expression does not affect the electric or magnetic fields, so we can choose a value arbitrarily. The value of $\nabla\cdot\mathbf{A}$ is known as the *gauge*, and the act of choosing the value is known as *gauge fixing*. There is a particular choice of gauge that makes the two equations maximally symmetric:

$$\nabla \cdot \mathbf{A} = -\frac{1}{c^2}\frac{\partial \phi}{\partial t}$$

which is known as the Lorentz gauge. Using this gauge condition, Maxwell's equations in potential form can be written as

$$\nabla^2 \phi - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = -\frac{\rho}{\epsilon_0}$$

$$\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \mathbf{J}$$

INTERPRETATION

The choice of gauge is often thought to be physically meaningless due to the fact that it does not affect the observables $\mathbf{E}$ and $\mathbf{B}$. Philosophically though, it is possible that the universe operates based on rules that assume a particular gauge. If so, then we could say philosophically that this gauge is the "correct" gauge choice, even though it may be impossible to tell which one it is. But even if there is no direct experimental test, there may be theoretical evidence that points toward a particular gauge over all the others. Perhaps one such piece of evidence is the observation that the Lorentz gauge equation is a continuity equation, fitting the general form

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \mathbf{j}$$

It is not surprising that ϕ obeys a continuity equation. A corollary of the Helmholtz Decomposition Theorem says that all physically realistic scalar fields obey a continuity equation. Basically, the theorem states that for any reasonable scalar field S and vector field $\mathbf{C}$ there exists a vector field $\mathbf{F}$ such that $\nabla \cdot \mathbf{F} = S$ and $\nabla \times \mathbf{F} = \mathbf{C}$. So if we choose $S = \frac{\partial \phi}{\partial t}$, the theorem implies the existence of a vector field $\mathbf{F}$ satisfying $\nabla \cdot \mathbf{F} = \frac{\partial \phi}{\partial t}$. Defining $\mathbf{A} = -\mathbf{F}/c^2$ gives us a familiar continuity equation: the Lorentz gauge condition. The interesting point is that the two potentials appear in a single continuity equation.

Continuity equations contain two variables: a density and a flux. Typically, the density corresponds to some physical quantity and the flux corresponds to the flow of that physical quantity e.g. charge and current. And in such cases, there is only one underlaying scalar field. For example, consider the continuity equation for probability in quantum mechanics, which only depends on the underlying wave function field; the probability flux density $\mathbf{j}$ is just a formula referring to ψ. This suggests that the electromagnetic potentials may likewise derive from a single scalar field.

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